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On classification of affine Kac–Moody groups[1]

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Abstract

It is shown that two affine Kac–Moody groups are isomorphic to each other if and only if their root systems, as well as the base fields, are isomorphic to each other. The analogous property for the related loop groups is also found to be true. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction and main theorem

The completion of the classification of Kac–Moody algebras has been done by Peterson and Kac in [10] with a demonstration of the isomorphism theorem which claims that two isomorphic Kac–Moody algebras must have, up to a permutation of the index, the same generalized Cartan matrix, or equivalently, their root systems isomorphic to each other. However, the classification for the groups associated to Kac–Moody algebras, the so-called Kac–Moody groups, remains unsolved. In this paper we demonstrate the isomorphism theorem for affine Kac–Moody groups as well as for the related loop groups. Let $\tilde{\mathfrak{g}}$ be a non-twisted affine Kac–Moody algebra over an algebraically closed field F of characteristic zero and let Φ be its real root system with respect to a fixed Cartan subalgebra. Denote by $\tilde{\mathfrak{g}}_a$ the root subspace of $\tilde{\mathfrak{g}}$ related to a root $a \in \Phi$. Let G^* be the free product of the additive groups $\tilde{\mathfrak{g}}_a$ for all $a \in \Phi$ and $\iota : \tilde{\mathfrak{g}}_a \to G^*$ the canonical embedding. Set $\tilde{\mathfrak{g}}' = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$. For any integrable $\tilde{\mathfrak{g}}'$-module V, or (V, ψ) with

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$\psi : \tilde{\mathfrak{g}}' \to End_F(V)$, let $\psi^* : G^* \to GL(V)$ be the homomorphism defined by

$$\psi^*(\imath(e)) = \exp \psi(e) \quad \text{for } a \in \Phi, \ e \in \tilde{\mathfrak{g}}_a,$$

where exp is the canonical exponential map. Suppose N is the intersection of all $\ker \psi^*$. The quotient group of G^* modulo N, denoted by $\tilde{G}(F)$, is called an affine (non-twisted) Kac–Moody group and the real root system Φ is also called a root system of $\tilde{G}(F)$. The original and other approaches of defining a Kac–Moody group can be found in [7, 9, 12]. The classification of Kac–Moody groups is basically the question: does an isomorphism between two Kac–Moody groups imply an isomorphism between their root system? The main purpose of this paper is to answer this question for affine Kac–Moody groups. The main result is as follows.

Theorem. *Let F and F' be algebraically closed fields of characteristic zero. Two non-twisted affine Kac–Moody groups $\tilde{G}(F)$ and $\tilde{G}'(F')$ are isomorphic to each other if and only if the following two properties hold*:
(i) *the root systems of $\tilde{G}(F)$ and $\tilde{G}'(F')$ are isomorphic to each other*;
(ii) *F is isomorphic to F'.*

One of the interests in the study of affine Kac–Moody groups comes from their close connection with loop groups. In fact, for showing the isomorphic property of affine Kac–Moody groups, we first demonstrate the isomorphism theorem for the related loop groups. It is shown that two such loop groups are isomorphic to each other if and only if they have isomorphic root systems as well as isomorphic base fields (see Lemma 2.2).

2. Proof of the theorem

We begin by recalling some basic properties of Chevalley groups over rings. In this paper the rings are always of characteristic zero. Let Φ_0 be a reduced irreducible root system of finite rank. From the root lattice or the weight lattice of Φ_0 one can construct a Chevalley–Demazure group scheme of adjoint type G_{ad} or of simply connected type G respectively (cf. [4, 5]). Those are covariant functors from the category of commutative rings with unity to the category of groups, in particular $G(F)$ (resp. $G_{\text{ad}}(F)$) is a simply connected (resp. adjoint) Chevalley group over an algebraically closed field F. Let R be a commutative ring with unity and R^+ be the additive group of R. Then for each root $a \in \Phi_0$ there exist canonical exponential homomorphisms of group: $u_a : R^+ \to G(R)$ and $\overline{u}_a : R^+ \to G_{\text{ad}}(R)$. The elementary subgroup of $G(R)$ (resp. $G_{\text{ad}}(R)$) is by definition the subgroup generated by $u_a(R)$ (resp. $\overline{u}_a(R)$) for all $a \in \Phi_0$.

Lemma 2.1. *Let G_{ad} and G'_{ad} be adjoint Chevalley–Demazure group schemes. Let R and R' be commutative integral domain containing $\mathbb{Q}$. If the elementary subgroups* *of $G_{ad}(R)$ and $G'_{ad}(R')$ are isomorphic to each other, then the root system of G_{ad} is isomorphic to that of G'_{ad} and R is isomorphic to R'.*

Proof. This comes from [3, Theorem 3.9]. □

A fundamental relation between a simply connected Chevalley–Demazure group scheme G and its adjoint type G_{ad} is that there exists an isogeny π from G to G_{ad} such that

$$\pi(u_a(r)) = \overline{u}_a(r) \quad \text{for } r \in R, \ \textit{for } a \in \Phi_0.$$

See [5] for a detailed description. We will use this relation to show the following property.

Lemma 2.2. *Let $F[t,t^{-1}]$ and $F'[s,s^{-1}]$ be Laurent polynomial rings over fields F and F' respectively. Let G and G' be simply connected Chevalley–Demazure group schemes. Then $G(F[t,t^{-1}])$ is isomorphic to $G'(F[s,s^{-1}])$ if and only if the following two properties hold*:

(i) *the root system of G and G' are isomorphic to each other*;

(ii) *F is isomorphic to F'.*

Proof. The availability of the property (i) implies that G and G' are isomorphic group schemes by the uniqueness theorem (cf. [5]), while the property (ii) implies that $F[t,t^{-1}]$ is isomorphic to $F'[s,s^{-1}]$. Consequently, $G(F[t,t^{-1}])$ is isomorphic to $G'(F'[s,s^{-1}])$ if both (i) and (ii) hold.

Conversely, suppose that the two groups $G(F[t,t^{-1}])$ and $G'(F'[s,s^{-1}])$ are isomorphic to each other. Let $\pi : G \to G_{ad}$ and $\pi' : G' \to G'_{ad}$ be the natural isogenies. Then we have

$$\pi(G(F[t,t^{-1}])) \subseteq G_{ad}(F[t,t^{-1}])$$

and

$$\pi'(G'(F'[s,s^{-1}])) \subseteq G'_{ad}(F'[s,s^{-1}]).$$

Moreover, let $Z(G(F[t,t^{-1}]))$ and $Z(G'(F'[s,s^{-1}]))$ be the centers of $G(F[t,t^{-1}])$ and $G'(F'[s,s^{-1}])$ respectively, we have

$$\operatorname{Ker}\pi \cap G(F[t,t^{-1}]) = Z(G(F[t,t^{-1}]))$$

and

$$\operatorname{Ker}\pi' \cap G'(F'[s,s^{-1}]) = Z(G'(F'[s,s^{-1}])).$$

Thus, we obtain

$$\begin{aligned}\pi(G(F[t,t^{-1}])) &\cong G(F[t,t^{-1}])/Z(G(F[t,t^{-1}])) \\ &\cong G'(F'[s,s^{-1}])/Z(G'(F'[s,s^{-1}])) \\ &\cong \pi'(G'(F'[s,s^{-1}])).\end{aligned}$$

It is known that a simply connected Chevalley group over a Euclidean domain is generated by its root subgroups (cf. [11, p. 115]). Let Φ_0 and Φ_0' be root systems of G and G' respectively. Then $G(F[t,t^{-1}])$ and $G'(F'[s,s^{-1}])$ are generated by their root subgroups $u_a(F[t,t^{-1}])$ $(a\in\Phi_0)$ and $u'_{a'}(F'[s,s^{-1}])$ $(a'\in\Phi_0')$ respectively, since a Laurent polynomial ring is also a Euclidean domain. Thus, $\pi(G(F[t,t^{-1}]))$ and $\pi'(G'(F'[s,s^{-1}]))$ are elementary subgroups of $G_{\mathrm{ad}}(F[t,t^{-1}])$ and $G'_{\mathrm{ad}}(F'[s,s^{-1}])$ respectively. It follows then from Lemma 2.1 that Φ_0 is isomorphic to Φ_0' and that $F[t,t^{-1}]$ is isomorphic to $F'[s,s^{-1}]$ as rings. Since F and F' are unique maximal subfields of $F[t,t^{-1}]$ and $F'[s,s^{-1}]$ respectively, they must be isomorphic to each other as required. $\square$

For demonstrating the isomorphism theorem of Kac–Moody groups, we need a description of their generators and relations. Let $\tilde{\mathfrak{g}}$ be an affine Kac–Moody algebra over F and $\tilde{A}$ be the generalized Cartan matrix associated to $\tilde{\mathfrak{g}}$, that is $\tilde{A}=(A_{ij})$, where $A_{ij}\in\mathbb{Z}$ for $i,j\in\{0,1,\dots,l\}$ such that

$$A_{ii}=2,\quad A_{ij}\le 0\quad \text{if } i\ne j,\quad A_{ij}=0\Leftrightarrow A_{ji}=0.$$

Let Φ be a real root system of $\tilde{\mathfrak{g}}$ and suppose $\Delta=\{a_0,a_1,\dots,a_l\}$ is a fundamental root system of Φ. We denote by W the Weyl group of Φ, which is generated by the fundamental reflections $\{s_0,s_1,\dots,s_l\}$ satisfying

$$s_i(a_j)=a_j-A_{ij}a_i,\qquad 0\le i,\ j\le l.$$

A pair of real roots $a,b\in\Phi$ is called prenilpotent if there exist elements $r,r'\in W$ such that

$$r(a),r(b)\in\left\{\sum_{i=0}^{l}n_ia_i\mid n_i\in\mathbb{N}\right\};\quad r'(a),r'(b)\in\left\{-\sum_{i=0}^{l}n_ia_i\mid n_i\in\mathbb{N}\right\}.$$

Let $ad:\tilde{\mathfrak{g}}\to End_F(\tilde{\mathfrak{g}})$ be the adjoint representation of $\tilde{\mathfrak{g}}$. For each $a\in\Phi$, it is possible to choose a base vector $\tilde{e}_a$ in the root subspace $\tilde{\mathfrak{g}}_a$ such that (see [6, Section 6])

$$\exp ad(\tilde{e}_{a_i})\cdot\exp ad(-\tilde{e}_{-a_i})\cdot\exp ad(\tilde{e}_{a_i})(e_a)=\tilde{e}_{s_i(a)}\ \text{or}\ -\tilde{e}_{s_i(a)},\qquad 0\le i\le l.$$

We define the number $n_{i,a}$ by

$$\exp ad(\tilde{e}_{a_i})\cdot\exp ad(-\tilde{e}_{-a_i})\cdot\exp ad(\tilde{e}_{a_i})(\tilde{e}_a)=n_{i,a}\tilde{e}_{s_i(a)}.$$

Let $\gamma:G^*\to\tilde{G}(F)$ be the natural homomorphism and, for each $a\in\Phi$, let $\lambda_a:F^+\to\tilde{\mathfrak{g}}_a$ be the homomorphism defined by

$$\lambda_a(q)=q\tilde{e}_a\quad\text{for } q\in F^+.$$

We write $\tilde{u}_a$ for the composite $\iota_a\gamma\lambda_a$ and define, for $q\in F^*$ and $i\in\{0,1,\dots,l\}$,

$$\tilde{w}_i(q)=\tilde{u}_{a_i}(q)\tilde{u}_{-a_i}(-q^{-1})\tilde{u}_{a_i}(q)$$

and

$$\tilde{h}_i(q) = \tilde{w}_i(q)\tilde{w}_i(1)^{-1}.$$

Lemma 2.3. *The Kac–Moody group $\tilde{G}(F)$ is generated by the elements $\tilde{u}_a(q)$ for $a \in \Phi$ and $q \in F$ subject to the following relations*:

(i) $\tilde{h}_i(p)\tilde{h}_i(q) = \tilde{h}_i(pq)$ *for* $p, q \in F^*$;

(ii) $\tilde{h}_i(p)\tilde{h}_j(q) = \tilde{h}_j(q)\tilde{h}_i(p)$ *for* $i, j \in \{0, 1, \dots, l\}$, $p, q \in F^*$;

(iii) $\tilde{u}_a(p)\tilde{u}_a(q) = \tilde{u}_a(p+q)$ *for* $a \in \Phi$, $p, q \in F$;

(iv) $\tilde{h}_j(q)\tilde{u}_{a_i}(p)\tilde{h}_j(q)^{-1} = \tilde{u}_{a_i}(q^{A_{ji}}p)$ *for* $p \in F$ *and* $q \in F^*$;

(v) $\tilde{w}_i(1)\tilde{h}_j(q)\tilde{w}_i(1)^{-1} = \tilde{h}_j(q)\tilde{h}_i(q^{-A_{ji}})$ *for* $q \in F^*$;

(vi) $\tilde{w}_i(1)\tilde{u}_a(p)\tilde{w}_i(1)^{-1} = \tilde{u}_{s_i(a)}(n_{i,a}p)$ *for* $a \in \Phi$ *and* $p \in F$;

(vii) *for every prenilpotent pair* $a, b \in \Phi$,

$$[\tilde{u}_a(p), \tilde{u}_F b(q)] = \prod_{\substack{i,j \in \mathbb{N}-\{0\} \\ ia+jb \in \Phi}} \tilde{u}_{ia+jb}(C_{ijab}p^i q^j), \quad p, q \in F,$$

where C_{ijab} is an integer determined uniquely by i, j, a, b and the order in which the terms on the right are taken.

Proof. It follows directly from the definition of $\tilde{G}(F)$ that $\{\tilde{u}_a(q) \mid q \in F, a \in \Phi\}$ is a set of generators of the group. The relations result from [12, 1]. □

Proof of the theorem. Let Φ and Φ' be the root systems of the affine Kac–Moody groups $\tilde{G}(F)$ and $\tilde{G}'(F')$ respectively. Suppose that Φ is isomorphic to Φ' and that F is isomorphic to F'. Let $\varphi : \Phi \to \Phi'$ be an isomorphism of the root systems and let $\varphi : F \to F'$ be an isomorphism of the fields. We define a map $\tilde{\alpha}$ between the generators of $\tilde{G}(F)$ and that of $\tilde{G}'(F')$ by

$$\tilde{\alpha}(\tilde{u}_a(q)) = \tilde{u}_{\varphi(a)}(\varphi(q)), \quad \textit{for } a \in \Phi,\ q \in F.$$

Thanks to Lemma 2.3, it is evident that $\tilde{\alpha}$ can be extended uniquely to an isomorphism between $\tilde{G}(F)$ and $\tilde{G}'(F')$.

Conversely, suppose that $\tilde{G}(F)$ is isomorphic to $\tilde{G}'(F')$. Note that Φ and Φ' are extensions of certain reduced root systems of finite type (cf. [8]), which we denoted by Φ_0 and Φ'_0 respectively. We will show that Φ_0 is isomorphic to Φ'_0, which implies immediately that Φ is isomorphic to Φ' since both of these root systems are assumed to be non-twisted affine root systems. Let G and G' be simply connected Chevalley–Demazure group schemes whose root systems are Φ_0 and Φ'_0 respectively. It is known (cf. [10]) that there are central extensions:

$$1 \to F^* \xrightarrow{\tau} \tilde{G}(F) \xrightarrow{\beta} G(F[t, t^{-1}]) \to 1$$

and

$$1 \to F'^* \xrightarrow{\tau'} \tilde{G}'(F') \xrightarrow{\beta'} G'(F'[s, s^{-1}]) \to 1,$$

where F^* and F'^* are the multiplicative groups of F and F' respectively. Let $\tilde{\alpha}:\tilde{G}(F)\to\tilde{G}'(F')$ be an isomorphism. We claim that

$$\tilde{\alpha}\tau(F^*)=\tau'(F'^*). \tag{1}$$

In fact, note that both $\tilde{\alpha}\tau(F^*)$ and $\tau'(F'^*)$ are contained in the center $Z(\tilde{G}(F))$ of $\tilde{G}'(F')$, we have a canonical isomorphism

$$\tilde{\alpha}\tau(F^*)/\tilde{\alpha}\tau(F^*)\cap\tau'(F'^*)\cong\tau'(F'^*)\tilde{\alpha}\tau(F^*)/\tau'(F'^*)\subseteq Z(G'(F'))/\tau'(F'^*).$$

Note that β' induces an isomorphism from $Z(G'(F'))/\tau'(F'^*)$ to the center of $G'(F'[s,s^{-1}])$, which is finite. Thus

$$|\tau'(F'^*)/\tilde{\alpha}\tau(F^*)\cap\tau'(F'^*)| < |Z(G'(F'))/\tau'(F'^*)| < \infty.$$

However, $\tau'(F'^*)$ is a divisible group since F' is algebraically closed, which means that $\tau'(F'^*)$ has no proper subgroup of finite index. This forces

$$\tilde{\alpha}\tau(F^*)\cap\tau'(F'^*)=\tau'(F'^*).$$

Hence $\tau'(F'^*)\subseteq\tilde{\alpha}(F^*)$. Taking $\tilde{\alpha}^{-1}$ instead of $\tilde{\alpha}$ and following a similar argument as above, one can obtain on the other hand that $\tilde{\alpha}\tau(F^*)\subseteq\tau'(F'^*)$. Thus, we have the identity (1).

Now, we define a map $\alpha:G(F[t,t^{-1}])\to G'(F'[s,s^{-1}])$ by

$$\alpha(g)=\beta'\cdot\tilde{\alpha}\cdot\beta^{-1}(g),\quad \text{for } g\in G(F[t,t^{-1}]).$$

This map is well defined because, for an arbitrary $g\in G(F[t,t^{-1}])$ and for two elements $x,y\in\beta^{-1}(g)$, we have $xy^{-1}\in\operatorname{Ker}\beta$, which means that $x=y\cdot\tau(q)$ for some $q\in F^*$ since $\operatorname{Ker}\beta=\tau(F^*)$, and therefore it comes from identity (1) that

$$\beta'\tilde{\alpha}(x)=\beta'\tilde{\alpha}(y\cdot\tau(q))=\beta'\tilde{\alpha}(y)\cdot\beta'\tilde{\alpha}\tau(q)=\beta'\tilde{\alpha}(y).$$

Thus, we have the following commutative diagram:

$$\begin{array}{ccc}
\tilde{G}(F) & \xrightarrow{\beta} & G(F[t,t^{-1}]) \\
\Big\downarrow{\scriptstyle\tilde{\alpha}} & & \Big\downarrow{\scriptstyle\alpha} \\
\tilde{G}'(F') & \xrightarrow{\beta'} & G'(F'[s,s^{-1}]).
\end{array}$$

It is obvious that α is a surjective homomorphism of group. We show that α is also injective. Suppose g is an element in $\operatorname{Ker}\alpha$. Again by identity (1) we have

$$\tilde{\alpha}\beta^{-1}(g)\in\operatorname{Ker}\beta'=\tilde{\alpha}\tau(F^*).$$

This implies that $\beta^{-1}(g) \in \tau(F^*)$. Hence $g \in \beta\tau(F^*) = \{1\}$. Thus, α is an isomorphism. Therefore, it follows from Lemma 2.2 (see also [2, Theorem 3.4]) that Φ_0 and Φ_0' are isomorphic to each other, from which we obtain that the property (i) of the theorem holds. Moreover, as a consequence of Lemma 2.2, the isomorphism α also yields the property (ii). This completes the proof. □

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